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How the Nonextensivity Parameter Affects Energy Fluctuations

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In this article, the effect of the nonextensivity parameter on the energy fluctuations of nonextensive systems is studied in two different versions of the Tsallis statistical mechanics. Once a general expression has been reported for the energy fluctuations in the second version (Tsallis work in 1988), the energy fluctuations of an ideal gas and a harmonic oscillator are studied in the second and fourth(OLM choice for the mean energy constrain) versions of the Tsallis statistical mechanics. The results for the fourth version indicate that relative energy fluctuations are strongly affected by the nonextensivity parameter via the number of accessible states. In fact, in the case of subextensive systems, the nonextensivity parameter leads to the fewer accessible states compared to that of the extensive systems and, therefore, smaller relative energy fluctuations are expected. For super-extensive systems, however, relative energy fluctuations are found to be larger than those in extensive systems because of the greater accessible states available. Our studies show that very large relative energy fluctuations are caused as the result of un-normalized nature of the second version which, in some cases, limits its application.

Keywords: Energy fluctuations, Nonextensive systems, Harmonic oscillator, Ideal gas, Tsallis statistical mechanics

INTRODUCTION

Generalized Tsallis statistical mechanics has, in recent years, received great attention by researchers working on such diverse fields as physics, chemistry, astrophysics, biology, and engineering. It has been used for predicting the phenomena associated with different nonextensive systems for which Boltzmann Gibbs (BG) statistics fails [1-11]. In this model, entropy is defined as [12]:

$$S = \frac{k}{q-1}\left(1 - \sum_{i}^{w} P_i^{\,q}\right) \qquad (1)$$

where, P_i is the probability of finding the system in the i^{th} state, k is a positive constant, and q is the entropic index related to the degree of nonextensivity.

Using different energy constraints in both normalized and un-normalized forms, various xpressions may be obtained for the probability and, consequently, for the partition function *via* maximizing the entropy function in the canonical ensemble. So far, four different energy constraints have been used that only one of which (known as the 2[nd] version) is un-normalized [5,12-13].

In this work, the number in different versions of Tsallis statistical mechanic refer to choices of the mean energy constrains. The first and second versions correspond to Tsallis work in 1988, original treatment, [12] and Curado-Tsallis choice of the mean energy constrain, respectively [5]. The third and fourth versions are according to Tsallis-Mendes-Plastino 1998 [6] and OLM [13] choices for the mean energy constrain, respectively. It is well established that there are reportedly certain unfamiliar consequences for the second or un-normalized versions of the Tsallis statistical mechanics [6]. In this work, we will investigate the effects of both the un-normalized nature of this version and the nonextensivity parameter on energy fluctuations. In simple model systems such as the free particle or the harmonic oscillator, energy fluctuations are strongly dependent on the version as well as the kind of the statistical mechanics in which the system is being investigated. One important aspect is the investigation of the spread of the

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probability distribution about the mean value, which is related to the fluctuations. This is because of the basic assumption in statistical mechanics that the ensemble average of a mechanical property is exactly equal to its experimental value. Moreover, the ensemble equivalence is established by investigating the fluctuations of different properties. A number of statistical thermodynamic theories are based on the fluctuations; for example, the formulation of the statistical mechanics for transport properties focuses on the decay rate of spontaneous fluctuations. Hence, studies are reported in the literature that have focused on energy fluctuations for an ideal gas and the harmonic oscillator in different versions of the Tsallis statistics [14-16]. For example, an expression was reported in 2003 for energy fluctuations in the first version of the Tsallis statistical mechanics in which the relative energy fluctuations consisted of two terms: the related heat capacity and the square of the mean energy. The relative energy fluctuations for a system with a large number of particles were reportedly small compared to those in the BG statistics [16].

For the third version, Liyan and Jiulin [15] showed that energy fluctuations for an ideal gas with a large number of particles would be negligible when $0 < q < 1$ and that the ensemble equivalence would, hence, be achieved in this region of the Tsallis statistical mechanics. The authors also reported the relative energy fluctuations related to $1/N$ in the nonextensive statistics rather than $1/\sqrt{N}$ in the extensive one [15].

Investigation of energy fluctuations for an ideal gas in both nonextensive (Tsallis) and extensive (Boltzmann-Gibbs) statistics have also revealed that the correlation induced by the nonextensivity of Tsallis entropy has an important role in energy fluctuations [14]. It has also been claimed that introducing the correlations between particles leads to smaller energy fluctuations [14].

It should be noted, however, that all these studies [14-16] were carried out on subextensive systems in which q is smaller than unity. Additionally, to the best of our knowledge, no investigation has been carried out on energy fluctuations in the second or fourth versions of the Tsallis statistics. While our aim in this article is to investigate the energy fluctuations in the second and fourth versions, the main question is whether the nonextensivity character of the system always leads to smaller fluctuations; more specifically, how are energy fluctuations dependent on the nonextensivity parameter?

To find an answer to this question, we first obtained the general expression for energy fluctuations in the second version of the Tsallis statistics. Although the simple ideal gas or harmonic oscillator models are not nonextensive systems [17], existence of the exact solution for these systems make their thermodynamic investigation useful to understand how the Tsallis formalism works. Therefore, the ideal gas and harmonic oscillator were investigated as simple textbook examples in the second and fourth versions of the Tsallis statistics, for q values smaller and larger than unity. It must be noted that the fourth version of the Tsallis statistical mechanics is a normalized version.

The remainder of this paper is organized as follows: In the next section, part *A*, energy fluctuations in the second version of the Tsallis statistics will be derived. In part *B*, energy fluctuations in the 2nd and 4th versions of the Tsallis statistical mechanics for an ideal gas and a harmonic oscillator will be discussed and compared with those in the Boltzmann-Gibbs statistical mechanics. Finally, conclusions will be presented.

RESULT AND DISCUSSION

Energy Fluctuations in the Second Version of Tsallis Statistical Mechanics

As already mentioned, the un-normalized and normalized energy constrains in the second and fourth versions of the Tsallis statistical mechanics have been used to obtain the probability in the canonical ensemble. Therefore, the different partition functions corresponding to these different constraints are:

$$\sum_i^w \varepsilon_i P_i^{\,q} - \overline{E_q} = 0\,;\; Z_q^{(2)} = \sum_i^w [1-(1-q)\beta\varepsilon_i]^{\frac{1}{(1-q)}},\; P_i = \frac{[1-(1-q)\beta\varepsilon_i]^{\frac{1}{(1-q)}}}{\sum_i^w [1-(1-q)\beta\varepsilon_i]^{\frac{1}{(1-q)}}}$$
$$\sum_i^w (\varepsilon_i - \overline{E_q}) P_i^{\,q} = 0\,;\; Z_q^{(4)} = \sum_i^w [1-(1-q)\beta(\varepsilon_i-\overline{E_q})]^{\frac{1}{(1-q)}},\; P_i = \frac{[1-(1-q)\beta(\varepsilon_i-\overline{E_q})]^{\frac{1}{(1-q)}}}{\sum_i^w [1-(1-q)\beta(\varepsilon_i-\overline{E_q})]^{\frac{1}{(1-q)}}} \quad (2)$$

where, superscripts denote the number of the statistics versions, ε_i represents the eigen values of the Hamiltonian of the system, $\overline{E_q}$ is the main energy of the system, $\beta = 1/kT$, T is the absolute temperature, and Z_q designates the generalized partition function. As is well known, all average values in the second version are un-normalized. Therefore, in this version, a different behavior is expected in certain characteristics of the system. For example, we know that the energy variance ($\overline{\Delta E^2}$) in the normalized statistical mechanics in the canonical ensemble is defined as [18]:

$$\overline{\Delta E^2} = \overline{(\bar{E} - E)^2} = \overline{E^2} - \overline{E}^2 \tag{3}$$

In fact, Eq. (3) may be applied in both the Boltzmann-Gibbs statistics and all normalized (1st, 3rd, and 4th) versions of the Tsallis statistics. In the second version, however, the square root of the energy variance is not equal to the energy fluctuations, and $\sqrt{\overline{(\bar{E} - E)^2}} \neq \sqrt{\overline{E^2} - \overline{E}^2}$ and $\overline{(\bar{E} - E)^2}$ can be calculated as follows:

$$\overline{\Delta E^2} = \overline{E^2} - 2\overline{E}^2 + \overline{E}^2 \sum_{i=0}^{w} P_i{}^q \neq \overline{E^2} - \overline{E}^2 \tag{4}$$

The inequality in Eq. (4) is due to $\langle 1 \rangle_q \neq 1$. In fact, in the second version of the Tsallis statistics, we have [5]:

$$\sum_i^w P_i{}^q = Z_q{}^{1-q} + (1-q)\beta\overline{E_q} \tag{5}$$

Therefore, the following result will be obtained for energy fluctuations:

$$\overline{\Delta E^2} = \overline{E^2} + \overline{E}^2 \left[Z_q{}^{1-q} - 2 + (1-q)\beta\overline{E_q}\right] \tag{6}$$

In the second version of the Tsallis statistical mechanics, $\overline{E_q{}^2}$ is defined as:

$$\overline{E_q{}^2} = \sum_i^w \varepsilon_i{}^2 \; P_i{}^q = -\frac{1}{Z_q{}^q}\frac{\partial}{\partial\beta}\left[\sum_i^w \varepsilon_i [1-(1-q)\beta\varepsilon_i]^{\frac{1}{(1-q)}}\right] \tag{7}$$

The term in the brackets may be rewritten as $Z_q \sum_i^w \varepsilon_i \; P_i$. Therefore, Eq. (7) can be rearranged as follows:

$$\overline{E_q{}^2} = \left[\overline{E_q}\left(\sum_1^w \varepsilon_i \; P_i\right) - \frac{1}{Z_q{}^{(q-1)}}\frac{\partial}{\partial\beta}\sum_i^w \varepsilon_i \; P_i\right] \tag{8}$$

Using the definition of the probability, *i.e.* Eq. (2), it will be possible to write $\sum_i^w \varepsilon_i \; P_i$ versus $\overline{E_q}$ and Z_q in this version as follows:

$$\sum_1^w \varepsilon_i \; P_i = \frac{1}{Z_q{}^{(1-q)}}\left[\sum_1^w \varepsilon_i \; P_i{}^q - (1-q)\beta\sum_1^w \varepsilon_i{}^2 \; P_i{}^q\right] = \frac{1}{Z_q{}^{(1-q)}}\left[\overline{E_q} - (1-q)\beta\overline{E_q{}^2}\right] \tag{9}$$

Inserting Eq. (9) into Eq. (8), we will have:

$$\overline{E_q{}^2} = \overline{E_q}\left(\frac{1}{Z_q{}^{(1-q)}}\left[\overline{E_q} - (1-q)\beta\overline{E_q{}^2}\right]\right) - \left(\frac{1}{Z_q{}^{(q-1)}}\frac{\partial}{\partial\beta}\left(\frac{\left[\overline{E_q} - (1-q)\beta\overline{E_q{}^2}\right]}{Z_q{}^{(1-q)}}\right)\right) \tag{10}$$

For the second term on the right hand side of the equation, we will have:

$$\frac{\partial}{\partial\beta}\left(\frac{\left[\overline{E_q} - (1-q)\beta\overline{E_q{}^2}\right]}{Z_q{}^{(1-q)}}\right) = \frac{Z_q{}^{(1-q)}\frac{\partial}{\partial\beta}\left[\overline{E_q} - (1-q)\beta\overline{E_q{}^2}\right] - \left[\overline{E_q} - (1-q)\beta\overline{E_q{}^2}\right]\frac{\partial}{\partial\beta}Z_q{}^{(1-q)}}{\left(Z_q{}^{(1-q)}\right)^2}) \tag{11}$$

and:

$$\frac{1}{Z_q{}^{(q-1)}}\frac{\partial}{\partial\beta}\left(\frac{\left[\overline{E_q} - (1-q)\beta\overline{E_q{}^2}\right]}{Z_q{}^{(1-q)}}\right) = \frac{\partial}{\partial\beta}\left[\overline{E_q} - (1-q)\beta\overline{E_q{}^2}\right] - \frac{1}{Z_q{}^{(1-q)}}\left[\overline{E_q} - (1-q)\beta\overline{E_q{}^2}\right]\frac{\partial}{\partial\beta}Z_q{}^{(1-q)} \tag{12}$$

Hence:

$$\overline{E_q{}^2} = \overline{E_q}\left(\frac{1}{Z_q{}^{(1-q)}}\left[\overline{E_q} - (1-q)\beta\overline{E_q{}^2}\right]\right) - \frac{\partial}{\partial\beta}\left[\overline{E_q} - (1-q)\beta\overline{E_q{}^2}\right] + \frac{1}{Z_q{}^{(1-q)}}\left[\overline{E_q} - (1-q)\beta\overline{E_q{}^2}\right]\frac{\partial}{\partial\beta}Z_q{}^{(1-q)} \quad (13)$$

and with proper simplifications, we may have:

$$\overline{E_q{}^2} = \frac{\overline{E_q}^2}{Z_q{}^{(1-q)}} - \frac{(1-q)\beta\overline{E_q}}{Z_q{}^{(1-q)}}\overline{E_q{}^2} - \frac{\partial\overline{E_q}}{\partial\beta} + (1-q)\beta\frac{\partial\overline{E_q{}^2}}{\partial\beta} + (1-q)\overline{E_q{}^2} + \frac{1}{Z_q{}^{(1-q)}}\left(\frac{\partial}{\partial\beta}Z_q{}^{(1-q)}\right)\left[\overline{E_q} - (1-q)\beta\overline{E_q{}^2}\right] \quad (14)$$

where, $\frac{\partial}{\partial\beta}Z_q{}^{(1-q)} = -(1-q)\overline{E_q}$; and finally:

$$\overline{E_q{}^2} = \frac{\overline{E_q}^2}{Z_q{}^{(1-q)}} - \frac{(1-q)\beta\overline{E_q}}{Z_q{}^{(1-q)}}\overline{E_q{}^2} - \frac{\partial\overline{E_q}}{\partial\beta} + (1-q)\beta\frac{\partial\overline{E_q{}^2}}{\partial\beta} + (1-q)\overline{E_q{}^2} - \frac{(1-q)\overline{E_q}^2}{Z_q{}^{(1-q)}} + \frac{(1-q)^2\beta\overline{E_q}}{Z_q{}^{(1-q)}}\overline{E_q{}^2} \quad (15)$$

Now, we subtract $(2-\sum_i^w P_i{}^q)\overline{E_q}^2$ from both sides of Eq. (15) to obtain:

$$\overline{\Delta E_q{}^2} = \frac{\overline{E_q}^2}{Z_q{}^{(1-q)}} - \frac{(1-q)\beta\overline{E_q}}{Z_q{}^{(1-q)}}\overline{E_q{}^2} - \frac{\partial\overline{E_q}}{\partial\beta} + (1-q)\beta\frac{\partial\overline{E_q{}^2}}{\partial\beta} + (1-q)\overline{E_q{}^2} - \frac{(1-q)\overline{E_q}^2}{Z_q{}^{(1-q)}} + \frac{(1-q)^2\beta\overline{E_q}}{Z_q{}^{(1-q)}}\overline{E_q{}^2} - \left(2-\sum_i^w P_i{}^q\right)\overline{E_q}^2 \quad (16)$$

And finally, the following Equation is obtained by rearranging Eq. (16):

$$\overline{\Delta E_q{}^2} = A\left(\overline{E_q{}^2} - \overline{E_q}^2\right) + \overline{E_q}^2(A-B) - \frac{\partial\overline{E_q}}{\partial\beta} + (1-q)\beta\frac{\partial\overline{E_q{}^2}}{\partial\beta} \quad (17)$$

where, the coefficients A and B are as:

$$A = (1-q) + \frac{(q^2-q)\beta\overline{E_q}}{Z_q{}^{(1-q)}}, \quad B = \left(2 + \beta(q-1)\overline{E_q} - Z_q{}^{(1-q)} - \frac{q}{Z_q{}^{(1-q)}}\right) \quad (18)$$

Therefore, the energy variance in the second version of the Tsallis statistics will be:

$$\overline{\Delta E_q{}^2} = -\frac{1}{1-A}\frac{\partial\overline{E_q}}{\partial\beta} + \frac{1-q}{1-A}\beta\frac{\partial\overline{E_q{}^2}}{\partial\beta} + \frac{A-B}{1-A}\overline{E_q}^2 \quad (19)$$

As shown in Eq.(19), the expression for the energy fluctuations in this version includes three terms. Comparison of Eq. (19) with the expressions for energy fluctuations in other versions of the Tsallis statistics [14-16] and also with that in BG [18] reveals that the first term on the right hand side of Eq.(19); *i.e.* the heat capacity term, is a common term in energy fluctuations for all the extensive and nonextensive systems of both normalized and un-normalized versions. The second term on the right hand side of Eq. (19) ($\frac{\partial\overline{E_q{}^2}}{\partial\beta}$) appears in the energy variance for nonextensive systems in the second and third versions of the Tsallis statistics. This term has been originated from the definitions of the average quantity in a weighted form, $P_i{}^q$. But the third term on the right hand side ($\overline{E_q}^2$) appears in the first and second versions as has been already reported by Potiguar and Costa for the first version [16].

It should be emphasized that the difference between the normalized and un-normalized versions arises from the fact that $\sum_{i=0}^w P_i{}^q \neq 1$, as is clear from Eq. (4). This un-normalized weighted probability, $\sum_{i=0}^w P_i{}^q \neq 1$, causes the coefficient B to appear in the third term of Eq. (19) .

When q→1 the coefficients A and B become zero and therefore Eq. (19) in the limit q→1 reduces to the corresponding equation in the Boltzmann Gibbs statistics:

$$\frac{\left(\overline{\Delta E_q{}^2}\right)^{\frac{1}{2}}}{\overline{E_q}} = -\frac{1}{\overline{E_q}}\left(\frac{\partial\overline{E_q}}{\partial\beta}\right)^{1/2} = \frac{1}{\overline{E_q}}(kT^2C_v)^{1/2} \quad (20)$$

In this work the relative energy variance and the relative energy fluctuations are defined as $\frac{\overline{\Delta E_q{}^2}}{\overline{E_q}}$ and $\frac{\left(\overline{\Delta E_q{}^2}\right)^{\frac{1}{2}}}{\overline{E_q}}$, respectively.

In the next section, the effect of the nonextensivity parameter on energy fluctuations in both normalized and un-normalized versions will be discussed in detail for the generalized ideal gas and harmonic oscillator cases. The obtained results will also be compared with the corresponding values in BG.

Energy Fluctuations for Generalized Ideal Gas and Harmonic Oscillator in the 2nd and 4th Versions of the Tsallis Statistics

The relative energy fluctuations for the ideal gas in the second version of the Tsallis statistical mechanics were calculated using Eq. (19). However, in the case of the ideal gas, analytical partition functions were available in some cases [19], all the data were obtained numerically because calculating $\overline{E_q{}^2}$ directly from the partition function is impossible. To obtain the required quantities, the following partition function was used:

$$Z_q = \int_0^{v_{max}} \phi(E)\left[1-\beta(1-q)\left(\frac{N}{2}mv^2\right)\right]^{\frac{1}{(1-q)}} dv \qquad (21)$$

where, $\Phi(E)$ is the degeneracy for an ideal gas velocity [18] and v_{max} is the maximum allowed velocity for the ideal gas molecules:

$$\Phi(E) = \frac{1}{\Gamma(N+1)\Gamma(3N/2)}\left(\frac{2\pi m a^2}{h^2}\right)^{3N/2} \qquad (22)$$

$$v_{max} = \begin{cases} \infty & q \geq 1 \\ \dfrac{2}{\sqrt{2\beta N m(1-q)}} & q < 1 \end{cases} \qquad (23)$$

where $a^3 = V$ which V is the volume.

In Fig. 1, the relative energy variance (REV) and its three parts are plotted versus q according to Eq. (19). These data were obtained for an argon molecule which is in a macroscopic enclosure at T = 300 K. The value of REV in the BG statistical mechanics, when q = 1, are also shown in this figure.

According to this figure, the relative energy fluctuations are usually greater than those in BG. In fact, only for a very small region around unity, when q is smaller than unity, the relative energy variance is less than that in BG. It should be noted that REV is used instead of relative energy fluctuations because some of the terms in Eq. (19) tend to be negative at times. However, the reported REV and relative energy fluctuations form two monotonic functions. The important point understood from this figure, is the domination of the third term of Eq. (19) in REV for different values of q except when $q \to 1^-$ (where REV is less than its corresponding value in BG). Since the un-normalized character of this version affects the values of this term, it suggests that the greater value of REV for the ideal gas, compared to its value in the BG, arises from the un-normalized nature of this version. To gain a better understanding of the role of the third term in REV, its value for a quantum harmonic oscillator in this version was also determined by using the partition function for a quantum harmonic oscillator in the second version, $Z_{vib,q}$, and Eq. (19):

$$Z_{vib,q} = \sum_{n=0}^{n_{max}}\left[1-\beta(1-q)\left(n+\frac{1}{2}\right)hv\right]^{\frac{1}{1-q}} \qquad (24)$$

where, v is the natural frequency of the oscillator and n_{max} is the quantum number that can be obtained as follows:

$$n_{max} = \begin{cases} \infty & q \geq 1 \\ integer\left(\dfrac{2-\beta hv(q-1)}{2\beta hv(1-q)}\right) & q < 1 \end{cases} \qquad (25)$$

It should be noted that for $q < 1$, the partition function can be obtained simply by direct summation over a finite number of states and that for $1 < q < 2$, an analytical expression can be obtained for the partition function [20]. However, as the case of the ideal gas, all the quantities in Eq. (19) for the harmonic oscillator were calculated numerically.

In Fig. 2, REV and its three parts as in Eq. (19) are plotted for the quantum harmonic oscillator and compared with the corresponding value in the BG statistics for $\beta hv = 1$.

Clearly, REV is smaller than that in BG when q is smaller than unity but greater than BG when $q > 1$. It should be noted that the anomalous behavior of REV for the harmonic oscillator when $q \approx 0.6$-0.7 is related to the number of accessible states, n_{max}, in this region. In fact, when q is smaller than unity, a positive probability will be gained in this version when the maximum quantum number is $n_{max} = \frac{2-\beta hv(q-1)}{2\beta hv(1-q)}$; therefore, for $\beta hv = 1$, n_{max} varies from 3 to 3.83 as a result of q changing from 0.6 to

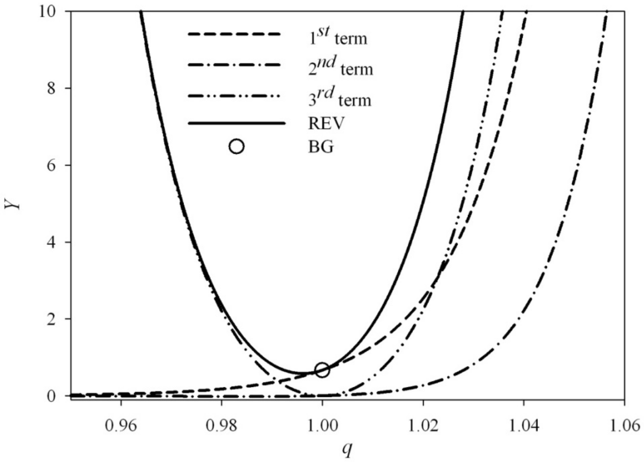

Fig. 1. The behavior of the three different terms on the right hand side of Eq. (19) and the relative energy variance (REV), *Y*, (all reduced to $\overline{E_q}^2$) *vs*. *q* for an argon molecule as an ideal gas at *T* = 300 K.

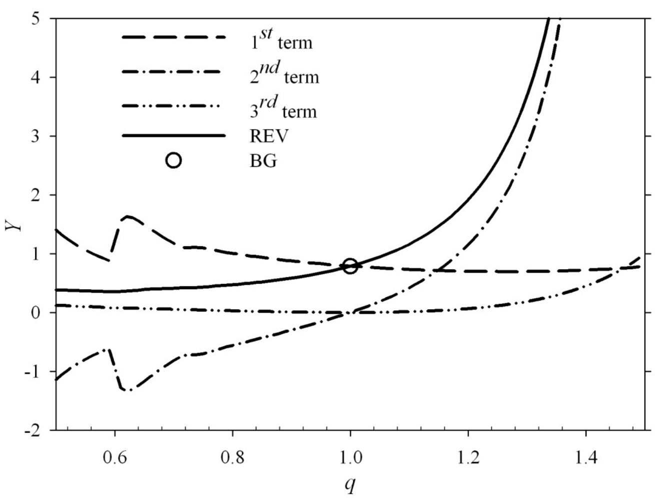

Fig. 2. The behavior of the three different terms on the right hand side of Eq. (19) and the relative energy variance (REV), *Y*, (all reduced to $\overline{E_q}^2$) *vs*. *q* for a quantum harmonic oscillator with *hv/kT* = 1.

0.7. Thus, the number of accessible states in this region will be 3. This limitation leads to the anomalous behavior of the first and second terms in Eq. (19) for the harmonic oscillator versus q, as shown in Fig. 2. Discontinuity in the mean energy versus *q* results in the irregular behavior of REV for the harmonic oscillator. Another point understood from this figure is that, contrary to the pervious case of the ideal gas, the third term does not play the main role in REV. To understand the conditions in which this term will have a dominant role in REV, the number of accessible states for the two cases of the ideal gas and harmonic oscillator were investigated. This is because it is a well established fact that energy fluctuations are strongly affected by the number of accessible states.

In Fig. 3a, the number of accessible states and the three dimensional velocity distribution of an argon atom at 300 K as an example of an ideal gas are plotted in both Tsallis and Boltzmann Gibs statistics. For this purpose, use was made of the following expression for the velocity distribution for an ideal gas:

$$P(v) = \frac{\left[1-\beta(1-q)\left(\frac{N}{2}mv^2\right)\right]^{\frac{1}{(1-q)}}}{Z_q} \tag{26}$$

where, Z_q is obtained from Eq. (21). It should be noted that in the case of the Tsallis statistics the distribution function is negative for some velocities when $q < 1$. This is physically not acceptable and, therefore, the distribution function has a cut-off. The cut-off velocity in the 2nd version of the Tsallis statistics was obtained using Eq. (23). As is clear from Fig. 3a, the velocity distribution is strongly dependent on the value of the entropic index, *q*, and the range of accessible velocities increases with increasing *q*.

Figure 3b displays the number of accessible states for a quantum harmonic oscillator in the 2nd version of the Tsallis statistics. The cut-off condition for a quantum harmonic oscillator in the 2nd version of Tsallis statistics is shown in Eq. (25). Again, the number of accessible states depends on *q* and increases with increasing *q*. However, it should be noted that the number of accessible states for the harmonic oscillator is much smaller than that for an ideal gas. It seems that the role of the third term in energy fluctuations is strongly dependent on both the number of accessible states and the shape of the probability. This term is, therefore, smaller for the harmonic oscillator than that for the ideal gas

Based on the above observations, for both ideal gas and harmonic oscillator, in the second version when q > 1, the value of REV is greater than that in BG. This is because, in

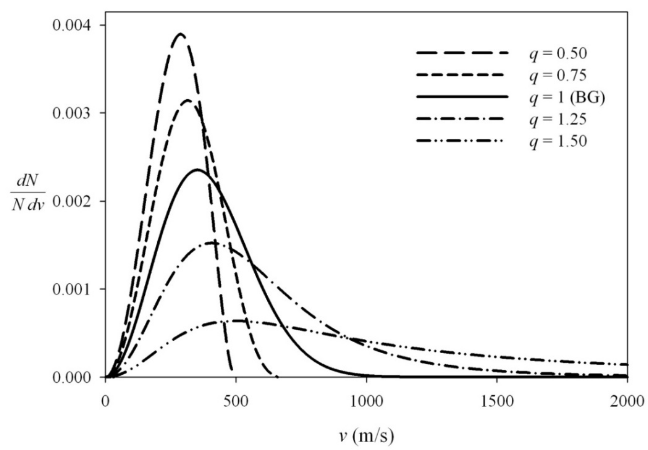

Fig. 3a. Comparison of the 3-dimensional velocity distribution of an Argon atom at 300K in extensive and nonextensive statistics (2nd version).

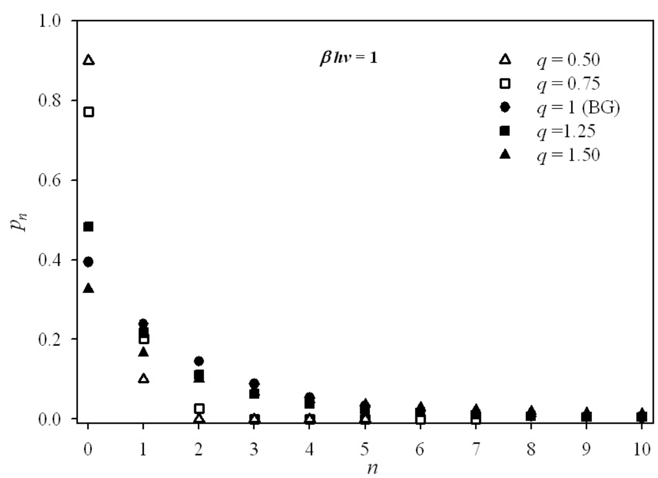

Fig. 3b. Comparison of the number of accessible states for a quantum harmonic oscillator in BG and in the 2nd version of the Tsallis statistics.

this region, all the three terms in Eq. (19) are involved in determining the value of REV. Based on Figs. 1 and 2, the behavior observed in this region does not originate only from the un-normalized picture of this version although it plays an important role. It is clear from Fig. 1, however, that the larger relative energy fluctuations for $q < 1$, compared to those in the Boltzmann Gibbs, are mainly due to the third term. In the case of the harmonic oscillator, the third term does not play any significant role in the energy fluctuations, because of the lower number of accessible states, and the REV value is, therefore, less than that in BG.

To further clarify the role of the nonextensivity parameter in REV and also to remove the effect of the un-normalized character of this version, the fourth version was investigated with respect to its REV value. In this version of the Tsallis statistics, the partition function may be obtained from the relation below:

$$Z_q = \int_0^{v_{max}} \phi(E)\left[1-\beta(1-q)\left(\frac{N}{2}mv^2-\overline{E_q}\right)\right]^{\frac{1}{(1-q)}} dv \qquad (27)$$

where, $\overline{E_q} = 3/2NkT$ is the mean value of energy for an ideal gas and its value is only dependent on temperature as is also the case in the BG statistics. v_{max} in Eq. (27) should be obtained from the following equation:

$$v_{max} = \begin{cases} \infty & q \geq 1 \\ \dfrac{2}{\sqrt{2\beta(1-\frac{3N}{2})Nm(1-q)}} & q < 1 \end{cases} \qquad (28)$$

In this manner, the partition function for a harmonic oscillator in the 4th version of the Tsallis statistics is as follows:

$$Z_q = \sum_{n=0}^{n_{max}} [1-(1-q)\beta((n+1/2)h\nu - \overline{E_q})]^{\frac{1}{(1-q)}} \qquad (29)$$

where, $\overline{E_q}$ is the mean value of the energy for the harmonic oscillator and n_{max} is obtained from the following equation:

$$n_{max} = \begin{cases} \infty & q \geq 1 \\ integer\left(\dfrac{2-(1-q)\beta(h\nu+2\overline{E_q})}{(1-q)\beta h\nu}\right) & q < 1 \end{cases} \qquad (30)$$

It is noteworthy that all the quantities in the 4th version must be calculated numerically and iteratively. However, in the case of an ideal gas, iteration is not needed in the calculations because mean energy is independent of the entropic index, q, and the partition function can be obtained directly. The results for the relative energy fluctuations in

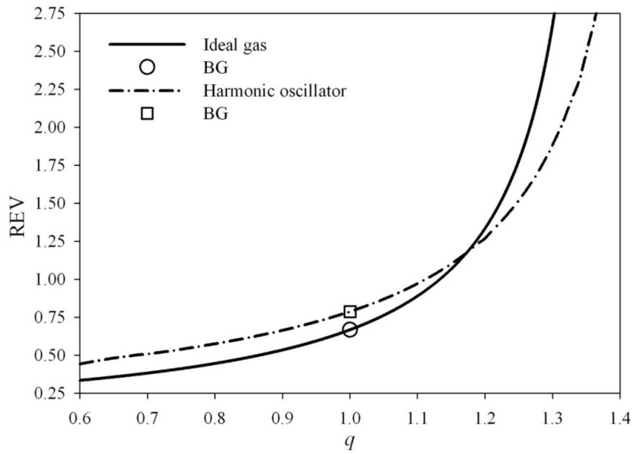

Fig. 4. Relative energy variance (REV) for an argon atom at 300 K and a quantum harmonic oscillator in the 4th version of the Tsallis statistics.

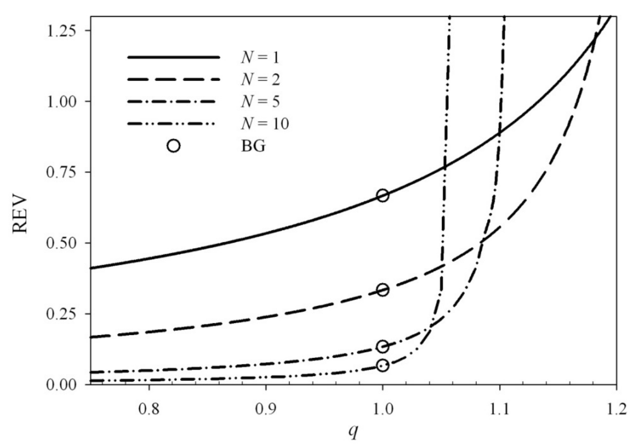

Fig. 5. Relative energy variance (REV) versus q for systems with different numbers of Argon atoms at 300 K in the 4th version of the Tsallis statistics.

the case of the argon atom in a macroscopic three dimensional box at T = 300 K for the 4th version are shown in Fig. 4.

As shown in this figure, the relative energy variance will be smaller than its corresponding BG value (*i.e.*, 2/3) when the entropic index, q, is smaller than unity and it will be greater than 2/3 when q is greater than 1.

The relative energy fluctuation for a quantum harmonic oscillator in the 4th version of the Tsallis statistics is also shown in the same figure. Clearly, the behavior of the relative energy variance for the harmonic oscillator is quite similar to that for the ideal gas. It may be concluded form Fig. 4 that relative energy fluctuations are strongly dependent on the value of the nonextensivity parameter via the number of accessible states. In fact, compared to the extensive system, the nonextensivity parameter in the case of subextensive systems causes a lower number of accessible states and, therefore, smaller relative energy fluctuations are expected. For super-extensive systems, however, relative energy fluctuations are greater than those in the extensive systems because of their greater number of accessible states.

One last point of great significance is the relationship between relative energy fluctuations and the number of system particles, N. In Fig. 5, the relative energy variance for an ideal gas with different numbers of particles is shown in the 4th version. As already mentioned in Introduction [15], the energy fluctuations in the 3rd version of the Tsallis statistics depends on $1/N$ rather than on $1/\sqrt{N}$ in the BG statistics. This means that there are fewer relative energy fluctuations in the case of nonextensive systems compared to extensive ones. According to Fig. 5, this is true only for $q < 1$, while for $q > 1$ as in one-particle systems, the relative energy fluctuations of a nonextensive system is always greater than those of an extensive one. It is interesting that the slope of variations in relative fluctuations with q increases with the number of particles in the case of a nonextensive system. This could be due to the increasing number of accessible states due to the dependence of degeneracy, $\Phi(E)$, on the number of system particles. It is worth noting that the entropic index, q, depends on the number of particles, N, for q>1 and that when N tends to infinity, q approaches unity [21]; therefore, no macroscopic ideal gas exists with $q \gg 1$.

CONCLUSIONS

In this article, the dependency of relative energy fluctuations on the entropic index has been studied for two different versions of Tsallis statistical mechanics; namely, the second and the fourth versions. It was demonstrated that the degree of relative energy fluctuations is strongly dependent on the version in which the systems are being investigated. The obtained results of the second version show that the un-normalized nature of this version essentially affects energy fluctuations. For the cases in which the number of accessible states is large, such as an ideal gas, the un-normalized picture of this version causes an anomalously large fluctuation. Our results also indicate an unfamiliar consequence of this version to the effect that the un-normalized weighted probability leads in some cases to large and anomalous fluctuations. Obviously, the results obtained for this version cannot be generalized to the other versions due to the un-normalized definition of average thermodynamic quantities. It was necessary, therefore, to further investigate the situation in a normalized version and to compare the results with the corresponding values in extensive systems. For this purpose, the fourth version was selected since the results in this version are capable of being generalized to non-extensive systems in the Tsallis statistical mechanics. Our investigations revealed that energy fluctuations are strongly dependent on the non-extensivity parameter via the number of accessible states such that the relative energy fluctuations were found to be smaller than those in BG when q was less than unity but they were greater when q was larger than unity. This result is also acceptable for systems with $N > 1$ particles. In fact, the energy fluctuations for sub-extensive systems are always smaller than those for extensive ones. However, this will be reversed when super-extensive systems are compared with extensive one. In other words, introducing a dynamic correlation between particles via the non-extensivity parameter leads to lower fluctuations in some cases (*i.e.*, $q < 1$) while it leads to large ones in others (*i.e.*, $q > 1$).

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REFERENCES

[1] J. Ruseckas, B. Kaulakys, Phys. Rev. E 84 (2011) 051125.
[2] D. Prato, C. Tsallis, Phys. Rev. E 60 (1999) 2398.
[3] S. Abe, Phys. Rev. E 79 (2009) 041116.
[4] M. Antoni, A. Torcini, Phys. Rev. E **5**7 (1998) R6233.
[5] E.M.F. Curado, C. Tsallis, J. Phys. A: Math. Gen 24 (1991) L69.
[6] C. Tsallis, R.S. Mendes, A.R. plastiono, Physica A 261 (1998) 534.
[7] J. Ruseckas, B. Kaulakys, Phys. Rev. E 81 (2010) 031105.
[8] S. Abe, C. Beck, E.G.D. Cohen, Phys. Rev. E 76 (2007) 031102.
[9] J.F. Lutsko, J.P. Boon, P. Grosfils, Europhys. Lett. 86 (2009) 40005.
[10] T. Oikonomou, G.B. Bagci, Phys. Lett. A 374 (2010) 2225.
[11] S. Abe, G.B. Bagci, Rev. E 71 (2005) 016139.
[12] C. Tsallis, J. Stat. Phys. 52 (1988) 479.
[13] G.L. Ferri, S. Martínez, A. Plastino, J. Stat. Mech. (2005) P04009.
[14] Z.H. Feng, L.Y. Liu, Physica A 389 (2010) 237.
[15] L. Liyan, D. Jiulin, Physica A 387 (2008) 5417.
[16] F.Q. Potiguar, U.M.S. Costa, Physica A 321 (2003) 482.
[17] G.B. Bagci, T. Oikonomou, Phys. Rev. E 88 (2013) 042126.
[18] D.A. McQuarrie, Statistical Mechanics, Harper Collins Publisher: Dunmore, PA, USA, 1976.
[19] A.R. Plastino, A. Plastino, C. Tsallis, J. Phys. A: Math. Gen. 27 (1994) 5707.
[20] E. Keshavarzi, M. Sabzehzari, M. Eliasi, Physica A 389 (2010) 2733.
[21] S. Abe, S. Martinez, F. Pennini, Phys. Lett. A 278 (2001) 249.